

A SIMPLY CONNECTED SURFACE OF GENERAL TYPE WITH $p_g = 0$ AND $K^2 = 3$

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ABSTRACT. Motivated by a recent result of Y. Lee and the second author [9], we construct a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$ using a rational blow-down surgery and \mathbb{Q} -Gorenstein smoothing theory. In a similar fashion, we also construct a new simply connected symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$.

1. INTRODUCTION

One of the fundamental problems in the classification of complex surfaces is to find a new family of simply connected surfaces of general type with $p_g = 0$. Although a large number of non-simply connected complex surfaces of general type with $p_g = 0$ have been known [5], simply connected surfaces of general type with $p_g = 0$ are little known except Barlow surface [4].

Recently, the second author constructed a simply connected symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 2$ using a rational blow-down surgery [19], and then Y. Lee and the second author constructed a family of simply connected, minimal, complex surface of general type with $p_g = 0$ and $1 \leq K^2 \leq 2$ by modifying Park's symplectic 4-manifold [9]. After this construction, it has been a natural question whether one can find a new family of surfaces of general type with $p_g = 0$ and $K^2 \geq 3$ using the same technique.

The aim of this article is to give an affirmative answer for the question above. Precisely, we are able to construct a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 3$ using a rational blow-down surgery and a \mathbb{Q} -Gorenstein smoothing theory developed in [9]. The key ingredient for the construction of $K^2 = 3$ case is to find a right rational surface Z which contains several disjoint chains of curves representing the resolution graphs of special quotient singularities. Once we have a right candidate Z for $K^2 = 3$, the remaining argument is parallel to that of $K^2 = 2$ case appeared in [9]. That is, we contract these chains of curves from the rational surface Z to produce a projective surface X with special quotient singularities. And then we prove that the singular surface X has a \mathbb{Q} -Gorenstein smoothing and the general fiber X_t of the \mathbb{Q} -Gorenstein smoothing is a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 3$. The main result of this article is the following.

Theorem 1.1. *There exists a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 3$.*

By using different pencils and fibrations, we provide more examples of simply connected minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 3$ in Section 6.

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Furthermore, we also construct a new simply connected closed symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$ using the same technique as in Section 3.

Theorem 1.2. *There exists a simply connected symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$ which is homeomorphic, but not diffeomorphic, to a rational surface $\mathbb{P}^2 \# 5\overline{\mathbb{P}}^2$.*

It is a very intriguing question whether the symplectic 4-manifold constructed in Theorem 1.2 above admits a complex structure. One way to approach this problem is to use \mathbb{Q} -Gorenstein smoothing theory as above. But, since the cohomology $H^2(T_X^0)$ is not zero in this case, it is hard to determine whether there exists a \mathbb{Q} -Gorenstein smoothing. Therefore we need to develop more \mathbb{Q} -Gorenstein smoothing theory in order to investigate the existence of a complex structure on the symplectic 4-manifold constructed in Theorem 1.2 above. We leave this question for future research.

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2. \mathbb{Q} -GORENSTEIN SMOOTHING

In this section we briefly review a theory of \mathbb{Q} -Gorenstein smoothing for projective surfaces with special quotient singularities and we quote some basic facts developed in [9].

Definition. Let X be a normal projective surface with quotient singularities. Let $\mathcal{X} \rightarrow \Delta$ (or \mathcal{X}/Δ) be a flat family of projective surfaces over a small disk Δ . The one-parameter family of surfaces $\mathcal{X} \rightarrow \Delta$ is called a *\mathbb{Q} -Gorenstein smoothing* of X if it satisfies the following three conditions;

- (i) the general fiber X_t is a smooth projective surface,
- (ii) the central fiber X_0 is X ,
- (iii) the canonical divisor $K_{\mathcal{X}/\Delta}$ is \mathbb{Q} -Cartier.

A \mathbb{Q} -Gorenstein smoothing for a germ of a quotient singularity $(X_0, 0)$ is defined similarly. A quotient singularity which admits a \mathbb{Q} -Gorenstein smoothing is called a *singularity of class T* .

Proposition 2.1 ([8, 12, 24]). *Let $(X_0, 0)$ be a germ of two dimensional quotient singularity. If $(X_0, 0)$ admits a \mathbb{Q} -Gorenstein smoothing over the disk, then $(X_0, 0)$ is either a rational double point or a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna - 1)$ for some integers a, n, d with a and n relatively prime.*

Proposition 2.2 ([8, 12, 25]). (1) *The singularities $\overset{-4}{\circ}$ and $\overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \dots - \overset{-2}{\circ} - \overset{-3}{\circ}$ are of class T .*

- (2) *If the singularity $\overset{-b_1}{\circ} - \dots - \overset{-b_r}{\circ}$ is of class T , then so are*

$$\overset{-2}{\circ} - \overset{-b_1}{\circ} - \dots - \overset{-b_{r-1}}{\circ} - \overset{-b_r-1}{\circ} \quad \text{and} \quad \overset{-b_1-1}{\circ} - \overset{-b_2}{\circ} - \dots - \overset{-b_r}{\circ} - \overset{-2}{\circ}.$$

- (3) *Every singularity of class T that is not a rational double point can be obtained by starting with one of the singularities described in (1) and iterating the steps described in (2).*

Let X be a normal projective surface with singularities of class T . Due to the result of Kollár and Shepherd-Barron [8], there is a \mathbb{Q} -Gorenstein smoothing locally for each singularity of class T on X . The natural question arises whether this local \mathbb{Q} -Gorenstein smoothing can be extended over the global surface X or not. Roughly geometric interpretation is the following: Let $\cup V_\alpha$ be an open covering of X such that each V_α has at most one singularity of class T . By the existence of a local \mathbb{Q} -Gorenstein smoothing, there is a \mathbb{Q} -Gorenstein smoothing $\mathcal{V}_\alpha/\Delta$. The question is if these families glue to a global one. The answer can be obtained by figuring out the obstruction map of the sheaves of deformation $T_X^i = \text{Ext}_X^i(\Omega_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. For example, if X is a smooth surface, then T_X^0 is the usual holomorphic tangent sheaf T_X and $T_X^1 = T_X^2 = 0$. By applying the standard result of deformations [10, 17] to a normal projective surface with quotient singularities, we get the following

Proposition 2.3 ([24], §4). *Let X be a normal projective surface with quotient singularities. Then*

- (1) *The first order deformation space of X is represented by the global Ext 1-group $\mathbb{T}_X^1 = \text{Ext}_X^1(\Omega_X, \mathcal{O}_X)$.*
- (2) *The obstruction lies in the global Ext 2-group $\mathbb{T}_X^2 = \text{Ext}_X^2(\Omega_X, \mathcal{O}_X)$.*

Furthermore, by applying the general result of local-global spectral sequence of ext sheaves ([17], §3) to deformation theory of surfaces with quotient singularities so that $E_2^{p,q} = H^p(T_X^q) \Rightarrow \mathbb{T}_X^{p+q}$, and by $H^j(T_X^i) = 0$ for $i, j \geq 1$, we also get

Proposition 2.4 ([12, 24]). *Let X be a normal projective surface with quotient singularities. Then*

- (1) *We have the exact sequence*

$$0 \rightarrow H^1(T_X^0) \rightarrow \mathbb{T}_X^1 \rightarrow \ker[H^0(T_X^1) \rightarrow H^2(T_X^0)] \rightarrow 0$$

where $H^1(T_X^0)$ represents the first order deformations of X for which the singularities remain locally a product.

- (2) *If $H^2(T_X^0) = 0$, every local deformation of the singularities may be globalized.*

The vanishing $H^2(T_X^0) = 0$ can be obtained via the vanishing of $H^2(T_V(-\log E))$, where V is the minimal resolution of X and E is the reduced exceptional divisors.

Theorem 2.1 ([9]). *Let X be a normal projective surface with singularities of class T . Let $\pi : V \rightarrow X$ be the minimal resolution and let E be the reduced exceptional divisors. Suppose that $H^2(T_V(-\log E)) = 0$. Then $H^2(T_X^0) = 0$ and there is a \mathbb{Q} -Gorenstein smoothing of X .*

Note that Theorem 2.1 above can be easily generalized to any log resolution of X by keeping the vanishing of cohomologies under blowing up at the points. It is obtained by the following well-known result.

Proposition 2.5 ([7], §1). *Let V be a nonsingular surface and let D be a simple normal crossing divisor in V . Let $f : V' \rightarrow V$ be a blowing up of V at a point p of D . Set $D' = f^{-1}(D)_{\text{red}}$. Then $h^2(T_{V'}(-\log D')) = h^2(T_V(-\log D))$.*

3. THE MAIN CONSTRUCTION

We begin with a special elliptic fibration $g : E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}}^2 \rightarrow \mathbb{P}^1$ which is constructed as follows: Let A be a line and B be a smooth conic in \mathbb{P}^2 such that A and B meet at two different points. Choose a tangent line L_1 to B at a point $p \in B$ so that L_1 intersects with A at a different point $q \in A$, and draw a tangent line L_2 from q to B which tangents at the point $r \in B$. Let L_3 be the line connecting p and r which meets A at s . We may assume that $p, r \notin A \cap B$ and $s \notin B$ (Figure 1).

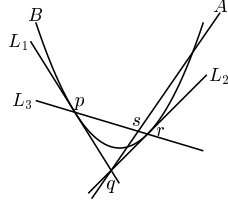
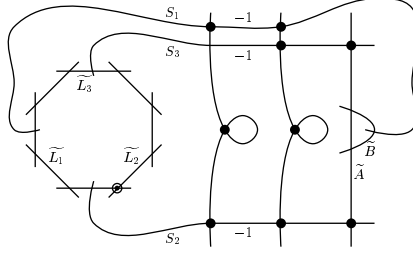


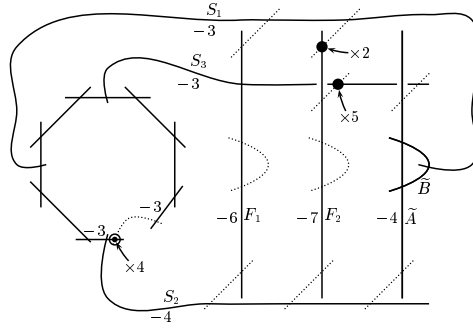
Figure 1: A cubic pencil

We now consider a cubic pencil in \mathbb{P}^2 induced by $A + B$ and $L_1 + L_2 + L_3$, i.e. $\lambda(A + B) + \mu(L_1 + L_2 + L_3)$, for $[\lambda : \mu] \in \mathbb{P}^1$. Blow up first at p and blow up at the intersection point of the proper transform of B with the exceptional curve e_1 . And then blow up again at the intersection point of the proper transform of B with the exceptional curve e_2 . Let e_3 be an exceptional divisor induced by the last blowing up. Similarly, after blowing up at r , blow up two more times at the intersection point of the proper transform of B with the exceptional curves e_4 and e_5 . Let e_6 be the exceptional divisor induced by the last blowing up. Next, blow up at q , and then blow up again at the intersection point of the proper transform of A with the exceptional curve e_7 . Let e_8 be the exceptional curve induced by the blowing up. Finally, blowing up once at s , which induces the exceptional divisor e_9 , we get an elliptic fibration $E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}}^2$ over \mathbb{P}^1 . Let us denote this elliptic fibration by $g : Y \rightarrow \mathbb{P}^1$. Note that there is an I_8 -singular fiber on $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of $L_1, L_2, L_3, e_1, e_2, e_4, e_5, e_7$. There is also one I_2 -singular fiber on $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of A and B , denoted by \tilde{A} and \tilde{B} respectively. According to the list of Persson [20], there exist only two more nodal singular fibers on $g : Y \rightarrow \mathbb{P}^1$. For example, the pencil used above can be chosen explicitly as follows: $\lambda x(x^2 + (y - 2z)^2 - z^2) + \mu(y - \sqrt{3}x)(y + \sqrt{3}x)(2y - 3z)$. Note that this pencil has singular fibers at $[\lambda : \mu] = [1 : 0], [0 : 1], [3\sqrt{3} : 2]$ and $[3\sqrt{3} : -2]$. Hence the fibration $g : Y \rightarrow \mathbb{P}^1$ has one I_8 -singular fiber, one reducible I_2 -singular fiber, and two nodal singular fibers. On the other hand, e_3, e_6, e_8 and e_9 in $Y = E(1)$ are the sections of $g : Y \rightarrow \mathbb{P}^1$ such that two sections e_3 and e_6 connect the proper transform \tilde{B} of B and the proper transforms of e_2, e_5 , respectively, and the other two sections e_8 and e_9 connect the proper transform \tilde{A} of A and the proper transforms of e_7, L_3 , respectively. Among these four sections, we will use only three sections, e_3, e_8 and e_9 , in the following main construction. We denote the three sections e_3, e_8, e_9 by S_1, S_2, S_3 , respectively (Figure 2).

Main Construction. Let $Z' := Y \# 2\overline{\mathbb{P}}^2$ be the surface obtained by blowing up at two singular points of two nodal fibers on Y , and denote this map by τ . Then there are two

Figure 2: An elliptic fibration $Y = E(1)$

fibers such that each consists of two \mathbb{P}^1 s, say E_i and F_i , satisfying $E_i^2 = -1$, $F_i^2 = -4$ and $E_i \cdot F_i = 2$ for $i = 1, 2$. Note that each E_i is an exceptional curve and F_i is the proper transform of a nodal fiber. We blow up twice at the intersection points between S_1 and F_i for $i = 1, 2$. We also blow up twice at the intersection points between S_3 and F_2 , \tilde{A} . And then blow up three times at the intersection points of between S_2 and F_1 , F_2 , \tilde{A} . Finally, blowing up at the marked point \odot on the I_8 -singular fiber, we then get $Z'' := Y \# 10\overline{\mathbb{P}}^2$ (Figure 3). Note that the self-intersection numbers of proper transforms are as follows: $[S_1]^2 = -3$, $[S_2]^2 = -4$, $[S_3]^2 = -3$, $[F_1]^2 = -6$, $[F_2]^2 = -7$ and $[\tilde{A}]^2 = -4$. Here we denote the proper transforms of S_i , F_j , \tilde{A} , \tilde{B} again by the same notations.

Figure 3: A rational surface $Z'' = Y \# 10\overline{\mathbb{P}}^2$

Next, we blow up two times successively at the intersection point between F_2 and the exceptional curve meeting F_2 and S_1 . It makes a chain of \mathbb{P}^1 , $\overset{-9}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transform of F_2 . We then blow up five times successively at the intersection point between S_3 and the exceptional curve meeting S_3 and F_2 , so that it produces a chain of \mathbb{P}^1 , $\overset{-8}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transform of S_3 . Finally, we blow up four times successively at the marked point \odot of the proper transform of the I_8 -singular fiber, so that it produces a chain of \mathbb{P}^1 , $\overset{-7}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$, lying in the proper transform of the I_8 -singular fiber. Note that the self-intersection numbers of proper transforms, denoted again by the same notations, are as follows: $[S_1]^2 = -3$, $[S_2]^2 = -4$, $[S_3]^2 = -8$, $[F_1]^2 = -6$, $[F_2]^2 = -9$ and $[\tilde{A}]^2 = -4$.

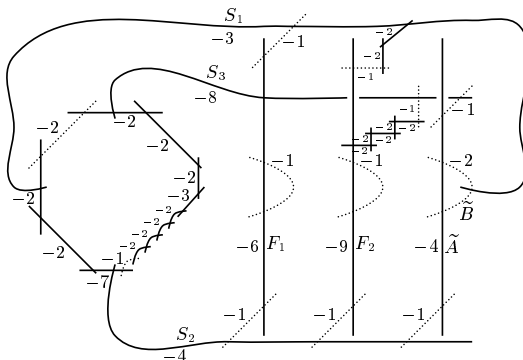


Figure 4: A rational surface $Z = Y \# 21\overline{\mathbb{P}}^2$

Finally, we follow the same procedures as in [9]. That is, we contract these four disjoint chains of \mathbb{P}^1 from Z . Since it satisfies the Artin's criterion, it produces a projective surface with four singularities of class T ([2], §2). We denote this surface by X . In the next sections, we are going to prove that X has a \mathbb{Q} -Gorenstein smoothing and a general fiber of the \mathbb{Q} -Gorenstein smoothing is a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$.

In the remaining of this section, we investigate a rational blow-down manifold of the surface Z obtained in the main construction above. First we describe topological aspects of a rational blow-down surgery ([6, 18] for details): For any relatively prime integers p and q with $p > q > 0$, we define a configuration $C_{p,q}$ as a smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere instructed by the following linear diagram

$$\begin{array}{ccccccc} -b_k & & -b_{k-1} & & \dots & & -b_2 & & -b_1 \\ \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ u_k & & u_{k-1} & & & & u_2 & & u_1 \end{array}$$

where $\frac{p^2}{pq-1} = [b_k, b_{k-1}, \dots, b_1]$ is the unique continued fraction with all $b_i \geq 2$, and each vertex u_i represents a disk bundle over the 2-sphere whose Euler number is $-b_i$. Orient the 2-spheres in $C_{p,q}$ so that $u_i \cdot u_{i+1} = +1$. Then the configuration $C_{p,q}$ is a negative definite simply connected smooth 4-manifold whose boundary is the lens space $L(p^2, 1 - pq)$.

Definition. Suppose M is a smooth 4-manifold containing a configuration $C_{p,q}$. Then we construct a new smooth 4-manifold M_p , called a (*generalized*) *rational blow-down* of M , by replacing $C_{p,q}$ with the rational ball $B_{p,q}$. Note that this process is well-defined, that is, a new smooth 4-manifold M_p is uniquely determined (up to diffeomorphism) from M because each diffeomorphism of $\partial B_{p,q}$ extends over the rational ball $B_{p,q}$. We call this a *rational blow-down surgery*. Furthermore, M. Symington proved that a rational

blow-down manifold M_p admits a symplectic structure in some cases. For example, if M is a symplectic 4-manifold containing a configuration $C_{p,q}$ such that all 2-spheres u_i in $C_{p,q}$ are symplectically embedded and intersect positively, then the rational blow-down manifold M_p also admits a symplectic structure [22, 23].

Now we perform a rational blow-down surgery of the surface Z obtained in the main construction. Note that the surface Z contains four disjoint configurations - $C_{35,6}$, $C_{19,5}$, $C_{7,1}$ and $C_{2,1}$. Let us decompose the surface Z into

$$Z = Z_0 \cup \{C_{35,6} \cup C_{19,5} \cup C_{7,1} \cup C_{2,1}\}.$$

Then the 4-manifold, say $Z_{35,19,7,2}$, obtained by rationally blowing down along the four configurations can be decomposed into

$$Z_{35,19,7,2} = Z_0 \cup \{B_{35,6} \cup B_{19,5} \cup B_{7,1} \cup B_{2,1}\},$$

where $B_{35,6}$, $B_{19,5}$, $B_{7,1}$ and $B_{2,1}$ are the corresponding rational balls. We claim that

Theorem 3.1. *The rational blow-down $Z_{35,19,7,2}$ of the surface Z in the main construction is a simply connected closed symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 3$.*

Proof. Since all the curves lying in the configurations $C_{35,6}$, $C_{19,5}$, $C_{7,1}$ and $C_{2,1}$ are symplectically (in fact, holomorphically) embedded 2-spheres, Symington's result [22, 23] guarantees the existence of a symplectic structure on the rational blow-down 4-manifold $Z_{35,19,7,2}$. Furthermore, it is easy to check that $b_2^+(Z_{35,19,7,2}) = b_2^+(Z) = 1$ and $K^2(Z_{35,19,7,2}) = K^2(Z) + 24 = 3$.

It remains to prove the simple connectivity of $Z_{35,19,7,2}$: Since $\pi_1(\partial B_{p,q}) \rightarrow \pi_1(B_{p,q})$ is surjective ([11], §5), by Van-Kampen theorem, it suffices to show that $\pi_1(Z_0) = 1$. First, note that Z and all four configurations $C_{35,6}$, $C_{19,5}$, $C_{7,1}$ and $C_{2,1}$ are all simply connected. Hence, applying Van-Kampen theorem on Z inductively, we get

$$(1) \quad 1 = \pi_1(Z_0) / \langle N_{i_*(\alpha)}, N_{j_{1*}(\beta_1)}, N_{j_{2*}(\beta_2)}, N_{j_{3*}(\beta_3)} \rangle.$$

Here i_* , j_{1*} , j_{2*} and j_{3*} are induced homomorphisms by inclusions $i : \partial C_{19,5} \rightarrow Z_0$, $j_1 : \partial C_{35,6} \rightarrow Z_0$, $j_2 : \partial C_{7,1} \rightarrow Z_0$ and $j_3 : \partial C_{2,1} \rightarrow Z_0$ respectively. We may also choose the generators, say α , β_1 , β_2 and β_3 , of $\pi_1(\partial C_{19,5}) \cong \mathbb{Z}_{19^2}$, $\pi_1(\partial C_{35,6}) \cong \mathbb{Z}_{35^2}$, $\pi_1(\partial C_{7,1}) \cong \mathbb{Z}_{7^2}$ and $\pi_1(\partial C_{2,1}) \cong \mathbb{Z}_{2^2}$, so that α , β_1 , β_2 and β_3 are represented by circles $\partial C_{19,5} \cap E'_1$ (equivalently $\partial C_{19,5} \cap E'_2$ or $\partial C_{19,5} \cap E'_3$), $\partial C_{35,6} \cap E'_1$, $\partial C_{7,1} \cap E'_2$ and $\partial C_{2,1} \cap E'_3$, respectively, where E'_1 , E'_2 and E'_3 are exceptional curves connecting the last 2-spheres in the configurations $C_{19,5}$ and $C_{35,6}$, $C_{19,5}$ and $C_{7,1}$, $C_{19,5}$ and $C_{2,1}$, respectively. Note that the circle cut out by a 2-sphere which intersects transversely one of the two end 2-spheres in the configurations $C_{p,q}$ is a generator of π_1 of the lens space, and other circles cut out by a 2-sphere which intersects transversely one of the middle 2-spheres in the configurations $C_{p,q}$, is a power of the generator [14]. Finally $N_{i_*(\alpha)}$ and $N_{j_{i*}(\beta_i)}$ denote the least normal subgroups of $\pi_1(Z_0)$ containing $i_*(\alpha)$ and $j_{i*}(\beta_i)$ respectively. Note that there is a relation between $i_*(\alpha)$ and $j_{i*}(\beta_i)$ when we restrict them to Z_0 . That is, they satisfy either $i_*(\alpha) = \gamma^{-1} \cdot j_{i*}(\beta_i) \cdot \gamma$ or $i_*(\alpha) = \gamma^{-1} \cdot j_{i*}(\beta_i)^{-1} \cdot \gamma$ (depending on orientations) for some path γ , because one is homotopic to the other in $E'_i \setminus \{\text{two open disks}\} \subset Z_0$. Hence, by combining two facts above, for example, we get $i_*(\alpha)^{19^2} = (\gamma^{-1} \cdot j_{1*}(\beta_1)^{\pm 1} \cdot \gamma)^{19^2} = \gamma^{-1} \cdot j_{1*}(\beta_1)^{\pm 19^2} \cdot \gamma = 1 = j_{1*}(\beta_1)^{35^2}$. Since the two numbers 19^2 and 35^2 are relatively prime, the element $j_{1*}(\beta_1)$ should be trivial. So the relation $i_*(\alpha) = \gamma^{-1} \cdot j_{1*}(\beta_1)^{\pm 1} \cdot \gamma$ implies the triviality of $i_*(\alpha)$.

Furthermore, since $i_*(\alpha)$ and $j_{i*}(\beta_i)$ are also conjugate to each other for $i = 2, 3$, the triviality of $i_*(\alpha)$ implies that $j_{2*}(\beta_2)$ and $j_{3*}(\beta_3)$ are trivial. Hence, all normal subgroups $N_{i_*(\alpha)}$ and $N_{j_{i*}(\beta_i)}$ are trivial, so that relation (1) implies $\pi_1(Z_0) = 1$. \square

4. EXISTENCE OF SMOOTHING

In this section we prove the existence of a \mathbb{Q} -Gorenstein smoothing for the singular projective surface X which is obtained by contracting from the rational surface Z in the main construction in Section 3. The procedure is exactly parallel to the $K^2 = 2$ case appeared in [9]. For the completeness of this article, we repeat the procedure here. First we need the following two essential lemmas.

Lemma 4.1 ([9]). *Let Y be a rational elliptic surface. Let C be a general fiber of the elliptic fibration $g : Y \rightarrow \mathbb{P}^1$. Then the global sections $H^0(Y, \Omega_Y(kC))$ are coming from the global sections $H^0(Y, g^*\Omega_{\mathbb{P}^1}(k))$. In particular, $h^0(Y, \Omega_Y(kC)) = k - 1$ for $k \geq 1$.*

Lemma 4.2. *Suppose $Z' = Y \# 2\overline{\mathbb{P}}^2$ is the rational elliptic surface in the main construction and F_i is the proper transform of a nodal fiber in Z' for $i = 1, 2$. Let \tilde{A} and \tilde{B} be the proper transforms of the line A and the conic B respectively. Let D be the reduced subscheme of the I_8 -singular fiber. Assume that D is not whole I_8 -singular fiber as a reduced scheme. Then $H^2(Z', T_{Z'}(-F_1 - F_2 - \tilde{A} - D)) = 0$.*

Proof. By the Serre duality, it is equal to prove $H^0(Z', \Omega_{Z'}(K_{Z'} + F_1 + F_2 + \tilde{A} + D)) = 0$. Let C be a general fiber in the elliptic fibration $g : Y \rightarrow \mathbb{P}^1$. Since $K_{Z'} = \tau^*(-C) + E_1 + E_2$ and $\tau^*(C) = F_1 + 2E_1 = F_2 + 2E_2$, $H^0(Z', \Omega_{Z'}(K_{Z'} + F_1 + F_2 + \tilde{A} + D)) \subseteq H^0(Z', \Omega_{Z'}(\tau^*(C) + \tilde{A} + D))$. Furthermore, since \tilde{A} and D are not changed by the map τ , we have the same curves in Y . Then $H^0(Z', \Omega_{Z'}(\tau^*(C) + \tilde{A} + D)) = H^0(Y, \Omega_Y(C + \tilde{A} + D))$ by the projection formula. We note that $\tau_*\Omega_{Z'} = \Omega_Y$. Then the cohomology $H^0(Y, \Omega_Y(C + \tilde{A} + D))$ vanishes: We note that $H^0(Y, \Omega_Y(C + \tilde{A} + D)) = H^0(Y, \Omega_Y(3C - \tilde{B} - G))$ with $G + D = I_8$ -singular fiber. By Lemma 4.1 above, all global sections of $\Omega_Y(3C)$ are coming from the global sections of $g^*(\Omega_{\mathbb{P}^1}(3)) = g^*(\mathcal{O}_{\mathbb{P}^1}(1))$. But, if this global section vanishes on \tilde{B} and G which lie on two different fibers, then it should be zero. Note that the dualizing sheaf of each fiber of the elliptic fibration is the structure sheaf of the fiber by using the adjunction formula. Therefore we have $H^2(Z', T_{Z'}(-F_1 - F_2 - \tilde{A} - D)) = 0$. \square

Theorem 4.1. *The projective surface X with four singularities of class T in the main construction has a \mathbb{Q} -Gorenstein smoothing.*

Proof. Let D be the reduced scheme of the I_8 -singular fiber minus the rational -2 -curve G in the main construction in Section 3. Note that the curve G is not contracted from Z to X . By Lemma 4.2 above, we have $H^2(Z', T_{Z'}(-F_1 - F_2 - \tilde{A} - D)) = H^2(Z', T_{Z'}(-\log(F_1 + F_2 + \tilde{A} + D))) = 0$. Let $D_{Z'} = F_1 + F_2 + \tilde{A} + D + S_1 + S_2 + S_3$. Since the self-intersection number of the section is -1 , we still have the vanishing $H^2(Z', T_{Z'}(-\log D_{Z'})) = 0$. Remind that the surface $Z'' = Y \# 10\overline{\mathbb{P}}^2$ is obtained by blowing up eight times from Z' : We blow up twice at the intersection points between S_1 and F_i for $i = 1, 2$. We also blow up twice at the intersection points between S_3 and F_2, \tilde{A} . And then blow up three times at the intersection points of between S_2 and F_1, F_2 ,

\tilde{A} . Finally, blowing up at the marked point \odot on the I_8 -singular fiber, we then get a rational surface Z'' (Figure 3). Now choose the exceptional curve in the total transform of I_8 -singular fiber which intersects the proper transform of I_8 , and choose two exceptional curves in the total transform of F_2 which intersect the proper transform of S_1 and S_3 . Let $D_{Z''}$ be the reduced scheme of $F_1 + F_2 + \tilde{A} + D + S_1 + S_2 + S_3 +$ these three exceptional divisors. Then, by Lemma 4.2, Proposition 2.5, and the self-intersection number, -1 , of each exceptional divisor, we have $H^2(Z'', T_{Z''}(-\log D_{Z''})) = 0$. Finally, by using the same argument finite times through blowing up, we have the vanishing $H^2(Z, T_Z(-\log D_Z)) = 0$, where D_Z are the four disjoint linear chains of \mathbb{P}^1 which are the exceptional divisors from the contraction from Z to X . Hence there is a \mathbb{Q} -Gorenstein smoothing for X by Theorem 2.1. \square

5. PROPERTIES OF X_t

We showed in Section 4 that the projective surface X has a \mathbb{Q} -Gorenstein smoothing. We denote a general fiber of the \mathbb{Q} -Gorenstein smoothing by X_t . In this section, we prove that X_t is a simply connected and minimal surface of general type with $p_g = 0$ and $K_{X_t}^2 = 3$ by using a standard argument. Of course, the procedure is also exactly parallel to the $K^2 = 2$ case appeared in [9].

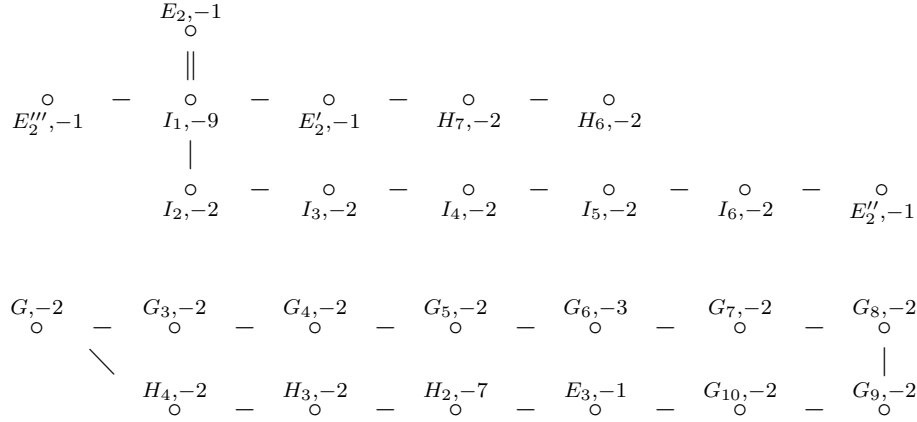
We first prove that X_t satisfies $p_g = 0$ and $K^2 = 3$: Since Z is a nonsingular rational surface and X has only rational singularities, X is a projective surface with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Then the upper semi-continuity implies that $H^2(X_t, \mathcal{O}_{X_t}) = 0$, so that the Serre duality implies that $p_g(X_t) = 0$. And $K_X^2 = 3$ can be computed by using the explicit description of f^*K_X (refer to Equation (2) below). Then we have $K_{X_t}^2 = 3$ by the property of the \mathbb{Q} -Gorenstein smoothing.

Next, let us show the minimality of X_t : As we noticed in Section 3, the surface Z contains the following four chains of \mathbb{P}^1 including the proper transforms of three sections. We denote them by the following dual graphs

$$\begin{aligned} C_{35,6} &= \begin{array}{cccccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 & G_8 & G_9 & G_{10} \end{array}, \\ C_{19,5} &= \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \end{array}, \\ C_{7,1} &= \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ I_1 & I_2 & I_3 & I_4 & I_5 & I_6 \end{array}, \quad C_{2,1} = \begin{array}{c} \circ \\ \tilde{A} \end{array} \end{aligned}$$

and we also denote the four special fibers by the following dual graphs:

$$\begin{array}{ccccc} \begin{array}{c} \circ \\ E_{4,-1} \end{array} & - & \begin{array}{c} \tilde{B}_{-2} \\ \circ \\ \parallel \\ \circ \\ \tilde{A}_{-4} \end{array} & - & \begin{array}{c} \circ \\ E'_{4,-1} \end{array} & & \begin{array}{c} \circ \\ E'_{1,-1} \end{array} & - & \begin{array}{c} \begin{array}{c} E_{1,-1} \\ \circ \\ \parallel \\ \circ \\ G_{1,-6} \end{array} & - & \begin{array}{c} \circ \\ E''_{1,-1} \end{array} \end{array}$$



Note that the final one indicates the total transform of I_8 -singular fiber and G denotes the rational -2 -curve which is not contracted from Z to X . The numbers indicate the self-intersection numbers of curves. Let $f : Z \rightarrow X$ and let $h : Z \rightarrow Y$. Then we have

$$K_Z \equiv f^* K_X - \left(\frac{29}{35} G_1 + \frac{34}{35} G_2 + \frac{33}{35} G_3 + \frac{32}{35} G_4 + \frac{31}{35} G_5 + \frac{30}{35} G_6 + \frac{24}{35} G_7 + \frac{18}{35} G_8 \right. \\ \left. + \frac{12}{35} G_9 + \frac{6}{35} G_{10} + \frac{14}{19} H_1 + \frac{18}{19} H_2 + \frac{17}{19} H_3 + \frac{16}{19} H_4 + \frac{15}{19} H_5 + \frac{10}{19} H_6 \right. \\ \left. + \frac{5}{19} H_7 + \frac{6}{7} I_1 + \frac{5}{7} I_2 + \frac{4}{7} I_3 + \frac{3}{7} I_4 + \frac{2}{7} I_5 + \frac{1}{7} I_6 + \frac{1}{2} \tilde{A} \right),$$

$$K_Z \equiv h^* K_Y + E_1 + E'_1 + E''_1 \\ + E_2 + 3E'_2 + 2H_7 + H_6 + 6E''_2 + 5I_6 + 4I_5 + 3I_4 + 2I_3 + I_2 + E'''_2 \\ + 5E_3 + 4G_{10} + 3G_9 + 2G_8 + G_7 \\ + E_4 + E'_4.$$

On the other hand, we have

$$h^* K_Y \equiv -\frac{1}{2} (2E_1 + E'_1 + E''_1 + G_1) \\ -\frac{1}{2} (2E_2 + 3E'_2 + 2H_7 + H_6 + 6E''_2 + 5I_6 + 4I_5 + 3I_4 + 2I_3 + I_2 + E'''_2 + I_1)$$

Hence, combining these relations, we get

$$(2) \quad f^* K_X \equiv \frac{1}{2} E'_1 + \frac{1}{2} E''_1 + \frac{3}{2} E'_2 + 3E''_2 + \frac{1}{2} E'''_2 + 5E_3 + E_4 + E'_4 + \frac{23}{70} G_1 + \frac{34}{35} G_2 \\ + \frac{33}{35} G_3 + \frac{32}{35} G_4 + \frac{31}{35} G_5 + \frac{6}{7} G_6 + \frac{59}{35} G_7 + \frac{88}{35} G_8 + \frac{117}{35} G_9 + \frac{146}{35} G_{10} \\ + \frac{14}{19} H_1 + \frac{18}{19} H_2 + \frac{17}{19} H_3 + \frac{16}{19} H_4 + \frac{15}{19} H_5 + \frac{39}{38} H_6 + \frac{24}{19} H_7 \\ + \frac{5}{14} I_1 + \frac{17}{14} I_2 + \frac{11}{7} I_3 + \frac{27}{14} I_4 + \frac{16}{7} I_5 + \frac{37}{14} I_6 + \frac{1}{2} \tilde{A}.$$

Since all the coefficients are positive in the expression of f^*K_X , the \mathbb{Q} -divisor f^*K_X is nef if $f^*K_X \cdot E_i \geq 0$ for $i = 3, 4$, and $f^*K_X \cdot E'_i \geq 0$ for $i = 1, 2, 4$, and $f^*K_X \cdot E''_i \geq 0$ for $i = 1, 2$, and $f^*K_X \cdot E'''_i \geq 0$. We have $f^*K_X \cdot E_3 = \frac{79}{665}$, $f^*K_X \cdot E_4 = \frac{33}{70}$, $f^*K_X \cdot E'_1 = \frac{411}{665}$, $f^*K_X \cdot E'_2 = \frac{16}{133}$, $f^*K_X \cdot E'_4 = \frac{9}{38}$, $f^*K_X \cdot E''_1 = \frac{376}{665}$, $f^*K_X \cdot E''_2 = \frac{43}{70}$, and $f^*K_X \cdot E'''_2 = \frac{79}{133}$. Note that other divisors are contracted under the map f . The nefness of f^*K_X implies the nefness of K_X . Since all coefficients are positive in the expression of f^*K_X , we get the vanishing $h^0(-K_X) = 0$. Hence, by the upper semi-continuity property, i.e. the vanishing $h^0(-K_X) = 0$ implies that $h^0(-K_{X_t}) = 0$, we conclude that X_t is not a rational surface: If X_t is a rational surface with $h^0(-K_{X_t}) = 0$, then $\chi(2K_{X_t}) \leq 0$. But $\chi(2K_{X_t}) = \chi(\mathcal{O}_{X_t}) + K_{X_t}^2 = 4$, which is a contradiction. Since $K_{X_t}^2 = 3$, X_t is a surface of general type by the classification theory of surfaces. Let $\pi : \mathcal{X} \rightarrow \Delta$ be a \mathbb{Q} -Gorenstein smoothing of X . Since the \mathbb{Q} -Cartier divisor $K_{\mathcal{X}/\Delta}$ is π -big over Δ and π -nef at the point 0, the nefness of K_{X_t} is also obtained by shrinking Δ if it is necessary [15]. Therefore we have

Proposition 5.1. *X_t is a minimal surface of general type with $p_g = 0$ and $K_{X_t}^2 = 3$.*

Finally, applying the standard arguments about Milnor fibers ([11], §5), we conclude that X_t is diffeomorphic to the rational blow-down 4-manifold $Z_{35,19,7,2}$ constructed in Theorem 3.1 (see [9] for details). Hence the simple connectivity of X_t follows from the fact that $Z_{35,19,7,2}$ is simply connected.

6. MORE EXAMPLES

In this section we construct another example of simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $K^2 = 3$ using a different configuration coming from a different elliptic pencil in \mathbb{P}^2 . Since all the proofs are basically the same as the case of the main example constructed in Section 3, we only explain how to construct it.

Construction. We first consider an elliptic fibration on $E(1)$ which has one I_6 -singular fiber, two I_2 -singular fibers, and two nodal fibers. Such an elliptic fibration can be constructed explicitly as follows: Let $A = \{[x : y : z] \in \mathbb{P}^2 : x + 2y + z = 0\}$ be a line and $B = \{[x : y : z] \in \mathbb{P}^2 : x^2 + xy + yz = 0\}$ be a conic in \mathbb{P}^2 . Let $L_1 = \{[x : y : z] \in \mathbb{P}^2 : z = 0\}$, $L_2 = \{[x : y : z] \in \mathbb{P}^2 : y = 0\}$, $L_3 := \{[x : y : z] \in \mathbb{P}^2 : x = 0\}$ be three coordinate lines. Denote the intersection points of $L_1 + L_2 + L_3$ and $A + B$ as follows (Figure 5):

$$\begin{aligned} p_1 &= [0 : 1 : 0] = B \cap L_1 \cap L_3, & p_2 &= [1 : -1 : 0] = B \cap L_1, \\ p_3 &= [2 : -1 : 0] = A \cap L_1, & p_4 &= [0 : 0 : 1] = B \cap L_2 \cap L_3, \\ p_5 &= [1 : 0 : -1] = A \cap L_2, & p_6 &= [0 : 1 : -2] = A \cap L_3. \end{aligned}$$

Note that B is tangent to L_2 at p_4 . We now consider a cubic pencil in \mathbb{P}^2 induced by $A+B$ and $L_1+L_2+L_3$, i.e., $\lambda(A+B) + \mu(L_1+L_2+L_3)$, for $[\lambda : \mu] \in \mathbb{P}^1$. This cubic pencil has a special member. If $\lambda = 1$ and $\mu = 2$, then the cubic pencil $\lambda(A+B) + \mu(L_1+L_2+L_3)$ is decomposed as follows:

$$(A + B) + 2(L_1 + L_2 + L_3) = (x + z)(x^2 + 3xy + 2y^2 + yz).$$

Set $C = \{[x : y : z] \in \mathbb{P}^2 : x + z = 0\}$ and $D = \{[x : y : z] \in \mathbb{P}^2 : x^2 + 3xy + 2y^2 + yz = 0\}$. It is obvious that C passes through p_1 and p_5 and D is a smooth conic in \mathbb{P}^2 tangent to L_2 at p_4 which also passes through p_2, p_3 , and p_6 (Figure 5).

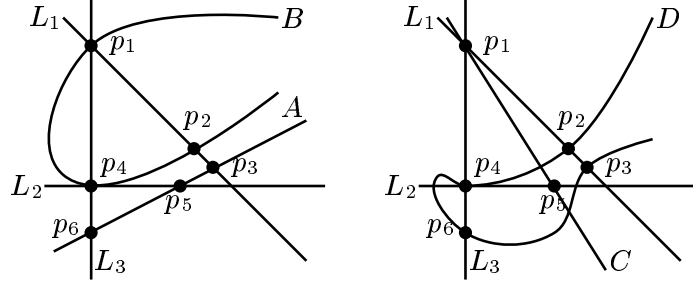
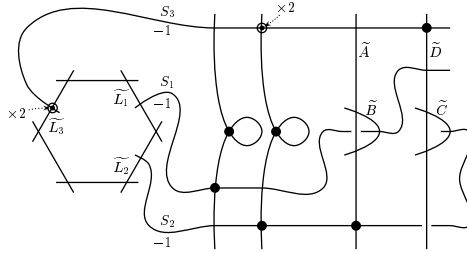


Figure 5: A cubic pencil

We next construct an elliptic fibration on $E(1)$ from the cubic pencil $\lambda(A + B) + \mu(L_1 + L_2 + L_3)$ as follows: Blow up at p_1 and blow up at the intersection point of the proper transform of B with the exceptional curve e_1 . Let e'_1 be an exceptional divisor induced by the last blowing up. Blow up at p_2, p_3, p_5 , and p_6 ; let e_2, e_3, e_5 , and e_6 be exceptional curves induced by the blowing ups, respectively. Finally, blow up at p_4 and blow up at the intersection point of the proper transform of B with the exceptional curve e_4 . And then blow up again at the intersection point of the proper transform of B with the exceptional curve e'_4 induced by the second blowing up. Let e''_4 be an exceptional divisor induced by the last blowing up. We get an elliptic fibration $E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}}^2$ over \mathbb{P}^1 . We denote this elliptic fibration by $g : Y = E(1) \rightarrow \mathbb{P}^1$ (Figure 6). Note that there is an I_6 -singular fibration on $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of L_1, L_2, L_3, e_1, e_4 , and e'_4 . There is also a I_2 -singular fiber on $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of A and B , denoted by \tilde{A} and \tilde{B} , respectively. Furthermore, there is another I_2 -singular fiber on $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of C and D , denoted by \tilde{C} and \tilde{D} . According to the list of Persson [20], there exist only two more nodal singular fibers. On the other hand, $e'_1, e_2, e_3, e'_4, e_5$, and e_6 in Y are the sections of $g : Y \rightarrow \mathbb{P}^1$. Among these sections, we use only e_2, e_5, e_6 in the following constructions. The section e_2 connects the proper transform of L_1 and \tilde{B}, \tilde{D} . The section e_5 connects the proper transform of L_2 and \tilde{A}, \tilde{C} . The section e_6 connects the proper transform of L_3 and \tilde{A}, \tilde{D} . We denote the three sections e_2, e_5, e_6 by S_1, S_2, S_3 , respectively (Figure 6).

Figure 6: An elliptic fibration $Y = E(1)$

Let $Z' := Y \# 2\overline{\mathbb{P}}^2$ be a rational surface obtained by blowing up at two singular points of two nodal fibers of Y . Let us denote the proper transforms of two nodal fibers by F_1 and F_2 . And we blow up once at the marked points \bullet on sections. We again blow up twice at the two marked points \odot on the section S_3 . Then we get a surface $Z := Y \# 10\overline{\mathbb{P}}^2$, which contains two disjoint linear chains of \mathbb{P}^1 : $C_{48,17} = \overset{-3}{\circ} - \overset{-6}{\circ} - \overset{-5}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ}$ (which contains the proper transforms of \tilde{A} , S_3 , F_1 , S_2 , a part of the I_6 -singular fiber), $C_{7,4} = \overset{-2}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ (which contains the proper transforms of F_2 , S_1 , \tilde{D}) (Figure 7).

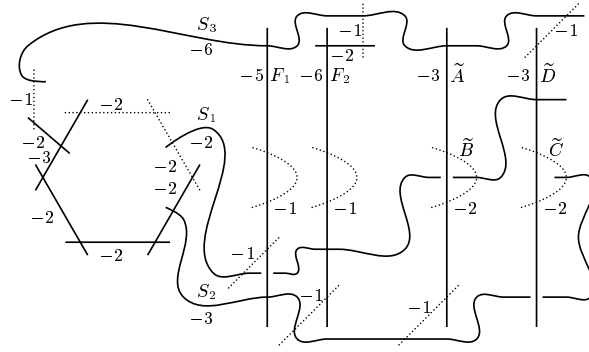


Figure 7: A rational surface $Z = Y \# 10\overline{\mathbb{P}}^2$

Finally, by contracting these two disjoint chains of \mathbb{P}^1 from Z , we obtain a projective surface X with two singularities of class T . The existence of a \mathbb{Q} -Gorenstein smoothing for X is obtained by the same proof as in Section 4. Let us denote a general fiber of the \mathbb{Q} -Gorenstein smoothing by X_t . Then, by the same argument in Section 5, we see that X_t is a minimal surface of general type with $p_g = 0$ and $K^3 = 3$. Furthermore, the rational blow-down 4-manifold $Z_{48,7}$ of the rational surface Z is simply connected, which can be proved in a similar way as in the proof of Theorem 3.1. Therefore we conclude that X_t is a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 3$.

Remark. One can find more surfaces using different configurations. For example, using an elliptic fibration on $E(1)$ which has one I_5 -singular fiber, one I_3 -singular fiber, one I_2 -singular fiber and two nodal fibers, we can construct another simply connected minimal surfaces of general type with $p_g = 0$ and $K^2 = 3$. But we do not know whether all these constructions above provide the same deformation equivalent type of surfaces with $p_g = 0$ and $K^2 = 3$. It is also a very interesting problem to determine whether these examples constructed above are diffeomorphic (or deformation equivalent) to the surface constructed in Section 3.

7. A SIMPLY CONNECTED SYMPLECTIC 4-MANIFOLD WITH $b_2^+ = 1$ AND $K^2 = 4$

In this section we construct a new simply connected symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$ using a rational blow-down surgery, and then we discuss the existence of a complex structure on it by using \mathbb{Q} -Gorenstein smoothing theory.

We first consider the elliptic fibration $g : Y = E(1) \rightarrow \mathbb{P}^1$ used in the main construction in Section 3, which has one I_8 -singular fiber, one I_2 -singular fiber, and three sections S_1 , S_2 , and S_3 . Among these sections, we use only S_1 and S_2 in the following construction (Figure 8).

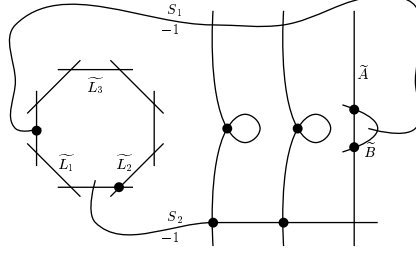


Figure 8: An elliptic fibration $Y = E(1)$

Let $Z' := Y \# 2\overline{\mathbb{P}}^2$ be a rational surface obtained by blowing up at two nodal points of two nodal singular fibers of Y . We denote the proper transforms of two nodal fibers by F_1 and F_2 . Blowing up once at the six marked points \bullet on S_2 , the I_8 -singular fiber and the I_2 -singular fiber, we get a rational surface $Z'' := Y \# 8\overline{\mathbb{P}}^2$ (Figure 9).

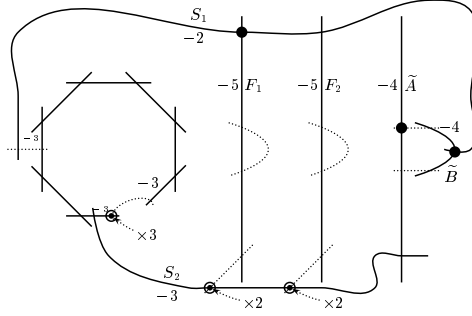
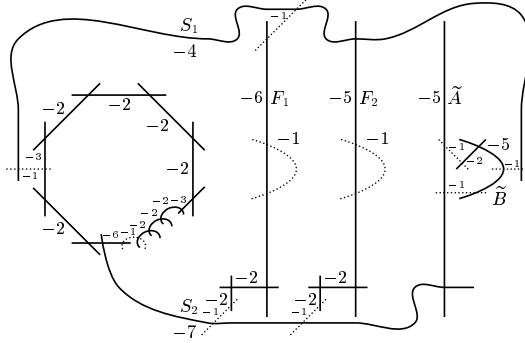


Figure 9: A rational surface $Z'' = Y \# 8\overline{\mathbb{P}}^2$

Finally, blow up once at the three marked points \bullet on the proper transforms of S_1 and \tilde{A} , and blow up again 3-times, 2-times, and 2-times as indicated in Figure 9 at the three marked points \odot on the proper transforms of I_9 -singular fibers and S_2 respectively. Then we get a rational surface $Z := Y \# 18\overline{\mathbb{P}}^2$, which contains four disjoint linear chains of \mathbb{P}^1 : $C_{131,27} = \overset{-5}{\circ} - \overset{-7}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which contains the proper transforms of \tilde{A} , S_2 , and the I_8 -singular fiber), $C_{7,2} = \overset{-4}{\circ} - \overset{-5}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which contains the proper transforms of S_1 and F_2), $C_{4,1} = \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which contains the proper transform of F_1), and $C_{3,1} = \overset{-5}{\circ} - \overset{-2}{\circ}$ (which contains the proper transform of \tilde{B}) (Figure 10).

Figure 10: A rational surface $Z = Y \# 18\overline{\mathbb{P}}^2$

Now we perform a rational blow-down surgery of the surface $Z = Y \# 18\overline{\mathbb{P}}^2$ as before. Note that the surface Z contains four disjoint configurations - $C_{131,27}$, $C_{7,2}$, $C_{4,1}$ and $C_{3,1}$. Let us decompose the surface Z into

$$Z = Z_0 \cup \{C_{131,27} \cup C_{7,2} \cup C_{4,1} \cup C_{3,1}\}.$$

Then the 4-manifold, say $Z_{131,7,4,3}$, obtained by rationally blowing down along the four configurations can be decomposed into

$$Z_{131,7,4,3} = Z_0 \cup \{B_{131,27} \cup B_{7,2} \cup B_{4,1} \cup B_{3,1}\},$$

where $B_{131,27}$, $B_{7,2}$, $B_{4,1}$ and $B_{3,1}$ are the corresponding rational balls. Then we have

Theorem 7.1. *The rational blow-down $Z_{131,7,4,3}$ of the surface Z in the construction above is a simply connected symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$ which is homeomorphic, but not diffeomorphic, to a rational surface $\mathbb{P}^2 \# 5\overline{\mathbb{P}}^2$.*

Proof. Since all proofs except the last statement are parallel to those of Theorem 3.1, we omit it. Furthermore, one can easily show that the standard canonical class K_Z of Z induces a non-trivial Seiberg-Witten basic class \widehat{K}_Z on $Z_{131,7,4,3}$ (refer to Theorem 3 in [19] or Theorem 3.1 in [21]). It means that $Z_{131,7,4,3}$ is not diffeomorphic to a rational surface $\mathbb{P}^2 \# 5\overline{\mathbb{P}}^2$. \square

Remarks. 1. In fact, one can prove that the symplectic 4-manifold $Z_{131,7,4,3}$ constructed in Theorem 7.1 above is minimal by using a technique in [16].

2. Recently, several authors constructed a simply connected minimal symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$ using a fiber-sum technique and a Luttinger surgery ([1, 3]). We do not know whether the symplectic 4-manifold $Z_{131,7,4,3}$ constructed in Theorem 7.1 above is diffeomorphic to one of their constructions. Despite of the fact, there is a big difference between our construction and theirs. That is, our example $Z_{131,7,4,3}$ has more room to investigate the existence of a complex structure, which is discussed below.

We close this article by discussing the possibility of the existence of a complex structure on the symplectic 4-manifold $Z_{131,7,4,3}$. One way to approach this problem is to use \mathbb{Q} -Gorenstein smoothing theory as in Section 4. That is, by contracting four disjoint chains of \mathbb{P}^1 from Z as before, we first obtain a projective surface X with four

singularities of class T . And then we investigate a \mathbb{Q} -Gorenstein smoothing for X . It is known that the cohomology $H^2(T_X^0)$ contains the obstruction space of a \mathbb{Q} -Gorenstein smoothing of X (refer to Proposition 2.4 and Theorem 2.1). That is, if $H^2(T_X^0) = 0$, then there is a \mathbb{Q} -Gorenstein smoothing of X . But the cohomology $H^2(T_X^0)$ is not zero in our case, so that it is hard to determine whether there exists a \mathbb{Q} -Gorenstein smoothing of X . Therefore we need to develop more \mathbb{Q} -Gorenstein smoothing theory in order to investigate the existence of a complex structure on the symplectic 4-manifold $Z_{131,7,4,3}$. We leave this question for future research.

Open Problem. Determine whether the symplectic 4-manifold $Z_{131,7,4,3}$ admits a complex structure.

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